Second form of adiabatic approximation.

Originally appeared at:
http://sites.google.com/site/peeterjoot2/math2011/adiabaticApproximation.pdf

Peeter Joot — peeter.joot@gmail.com

Dec 11, 2011 *adiabaticApproximation.tex*

Contents

1. Motivation.

In class we were shown an adiabatic approximation where we started with (or worked our way towards) a representation of the form

$$\left|\psi\right\rangle = \sum_k c_k(t) e^{-i \int_0^t (\omega_k(t') - \Gamma_k(t')) dt'} \left|\psi_k(t)\right\rangle \tag{1}$$

where $\left|\psi_k(t)\right\rangle$ were normalized energy eigenkets for the (slowly) evolving Hamiltonian

$$H(t) \left|\psi_k(t)\right\rangle = E_k(t) \left|\psi_k(t)\right\rangle \tag{2}$$

In the problem sets we were shown a different adiabatic approximation, where are starting point is

$$\left|\psi(t)\right\rangle = \sum_k c_k(t) \left|\psi_k(t)\right\rangle. \tag{3}$$

For completeness, here's a walk through of the general amplitude derivation that's been used.

2. Guts

We operate with our energy identity once again

$$\begin{aligned}
0 &= \left(H - i \hbar \frac{d}{dt} \right) \sum_k c_k \left|k\right\rangle \\
&= \sum_k c_k E_k \left|k\right\rangle - i \hbar c_k' \left|k\right\rangle - i \hbar c_k \left|k'\right\rangle,
\end{aligned}$$

where

$$\left|k'\right\rangle = \frac{d}{dt} \left|k\right\rangle. \tag{4}$$

Bra'ing with $\left\langle m\right|$, and split the sum into $k = m$ and $k \ne m$ parts

$$0 = c_m E_m - i\hbar c_m' - i\hbar c_m \left\langle m | m' \right\rangle - i\hbar \sum_{k \neq m} c_k \left\langle m | k' \right\rangle \tag{5}$$

Again writing

$$\Gamma_m = i \left\langle m | m' \right\rangle \tag{6}$$

We have

$$c_m' = \frac{1}{i\hbar} c_m (E_m - \hbar \Gamma_m) - \sum_{k \neq m} c_k \left\langle m | k' \right\rangle, \tag{7}$$

In this form we can make an "Adiabatic" approximation, dropping the $k \neq m$ terms, and integrate

$$\int \frac{dc_m'}{c_m} = \frac{1}{i\hbar} \int_0^t (E_m(t') - \hbar \Gamma_m(t')) dt' \tag{8}$$

or

$$c_m(t) = A \exp\left(\frac{1}{i\hbar} \int_0^t (E_m(t') - \hbar \Gamma_m(t')) dt' \right). \tag{9}$$

Evaluating at $t = 0$, fixes the integration constant for

$$c_m(t) = c_m(0) \exp\left(\frac{1}{i\hbar} \int_0^t (E_m(t') - \hbar \Gamma_m(t')) dt' \right). \tag{10}$$

Observe that this is very close to the starting point of the adiabatic approximation we performed in class since we end up with

$$\left| \psi \right\rangle = \sum_k c_k(0) e^{-i \int_0^t (\omega_k(t') - \Gamma_k(t')) dt'} \left| k(t) \right\rangle, \tag{11}$$

So, to perform the more detailed approximation, that started with 1, where we ended up with all the cross terms that had both ω_k and Berry phase Γ_k dependence, we have only to generalize by replacing $c_k(0)$ with $c_k(t)$.